Directed graphs and Maltsev conditions

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Digraphs, homomorphisms and polymorphisms

Definition

A **digraph** is a pair $\mathbb{G} = (G; \to)$, where G is the set of **vertices** and $\to \subseteq G^2$ is the set of **edges**.

Definition

A **homomorphism** from $\mathbb{G}$ to $\mathbb{H}$ is a map $f : G \to H$ that preserves edges:

$$a \to b \text{ in } \mathbb{G} \quad \Longrightarrow \quad f(a) \to f(b) \text{ in } \mathbb{H}.$$

Definition

A **polymorphism** of $\mathbb{G}$ is a homomorphism $p : \mathbb{G}^n \to \mathbb{G}$, that is, it preserves edges:

$$a_1 \to b_1, \dots, a_n \to b_n \implies p(a_1, \dots, a_n) \to p(b_1, \dots, b_n).$$

$\mathrm{Pol}(\mathbb{G}) = \{\, p \mid p : \mathbb{G}^n \to \mathbb{G} \,\}$ is the **clone of polymorphisms**.

Maltsev implies majority

Theorem (A. Kazda)

If a finite digraph has Maltsev polymorphism $p(x, y, y) = p(y, y, x) = x$*, then it has a majority polymorphism* $m(y, x, x) = m(x, y, x) = m(x, x, y) = x$.

Not true for finite relational structures.

Theorem (E. Aichinger, R. McKenzie, P. Mayer)

Every algebra with an edge-term is finitely related

$$
\begin{aligned}
p(y, y, x, x, \dots, x) &\approx x, \\
p(x, y, y, x, \dots, x) &\approx x, \\
p(x, x, x, y, \dots, x) &\approx x, \\
&\vdots \\
p(x, x, x, x, \dots, y) &\approx x.
\end{aligned}
$$

Jónsson implies near-unanimity

Theorem (L. Barto)

If a finite relational structure has Jónsson polymorphisms

$$\begin{aligned} x &= d_0(x, y, z), \\ d_i(x, y, x) &= x \text{ for all } i, \\ d_i(x, y, y) &= d_{i+1}(x, y, y) \text{ for even } i, \\ d_i(x, x, y) &= d_{i+1}(x, x, y) \text{ for odd } i, \\ d_n(x, y, z) &= z, \end{aligned}$$

then it has a near-unanimity polymorphism

$$t(y, x, \dots, x) = t(x, \dots, x, y) = x.$$

Valeriote's Conjecture

If a finite relational structure has Gumm polymorphisms, then it has an edge polymorphism.

Structure and Maltsev conditions

Theorem (L. Barto, M. Kozik)

If $\mathbb{G} = (G; E)$ is connected, $E \leq G^2$ is subdirect (smooth digraph), the algebraic length of $\mathbb{G}$ is 1, and it has a weak near-unanimity polymorphism, then $\mathbb{G}$ contains a loop.

Theorem (B. Larose, L. Zádori)

If a finite poset has Gumm polymorphisms

$$\begin{aligned} x &= d_0(x,y,z), \\ d_i(x,y,x) &= x \text{ for all } i, \\ d_i(x,y,y) &= d_{i+1}(x,y,y) \text{ for even } i, \\ d_i(x,x,y) &= d_{i+1}(x,x,y) \text{ for odd } i, \\ d_n(x,y,y) &= p(x,y,y), \text{ and} \\ p(x,x,y) &= y, \end{aligned}$$

then it has a near-unanimity polymorphism.

Reflexivity and Gumm implies near-unanimity

Theorem (B. Larose, C. Loten, L. Zádori)

If a finite symmetric reflexive digraph has Gumm polymorphisms, then it has a near-unanimity polymorphism.

Theorem

If a finite reflexive digraph $\mathbb{G}$ has Gumm polymorphisms then it has Jónsson (and near-unanimity) polymorphisms, and totally symmetric polymorphisms

$$\{a_1, \ldots, a_n\} = \{b_1, \ldots, b_n\} \implies t(a_1, \ldots, a_n) = t(b_1, \ldots, b_n)$$

of all arities.

Theorem

If a finite symmetric digraph $\mathbb{G}$ has Gumm polymorphisms then it has Jónsson (and near-unanimity) polymorphisms.

Structure on $\mathbb{G}^{\mathbb{H}}$

Definition

Let $\mathbb{G}$, $\mathbb{H}$ be digraphs and $f, g \in H^G$ be maps. We write $f \to g$ iff

$$a \to b \text{ in } \mathbb{G} \implies f(a) \to g(b) \text{ in } \mathbb{H}.$$

Lemma

- *The set of homomorphisms from $\mathbb{G}$ to $\mathbb{H}$ is*

$$\mathbb{H}^{\mathbb{G}} = \{ f \in H^G \mid f \to f \}.$$

- *If G is reflexive, then the Cartesian power of $\mathbb{G}$ is*

$$\mathbb{G}^n = \mathbb{G}^{\{\circlearrowleft \ \circlearrowleft \ \cdots \ \circlearrowleft\}}.$$

- *If $f \to g$ in $\mathbb{H}^{\mathbb{G}^n}$ and $f_1 \to g_1, \dots, f_n \to g_n$ in $\mathbb{G}^{\mathbb{F}}$, then*

$$f(f_1, \dots, f_n) \to g(g_1, \dots g_n) \textit{ in } \mathbb{H}^{\mathbb{F}}.$$

Connectedness on $\mathbb{G}^{\mathbb{G}}$

Theorem

Let $\mathbb{G}$ be a finite reflexive digraph admitting Gumm operations. If $\mathbb{G}$ is [strongly] connected, then so is $\mathbb{G}^{\mathbb{G}}$.

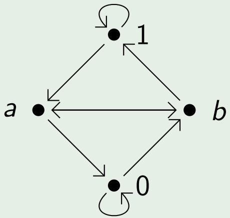
Proof.

- Take a minimal counterexample $\mathbb{G}$.
- $\{\mathrm{id}\}$ is a [strong] component of $\mathbb{G}^{\mathbb{G}}$.
- If $\mathbb{G}$ admits a ternary operation d satisfying
  - $d(x, y, y) \approx x$, or
  - $d(x, y, x) \approx d(x, x, y) \approx x$,

  then $d(x, y, z)$ is the first projection.
- Use the Gumm identities (or Hobby-McKenzie operations for omitting types **1** and **5**) to show that $\mathbb{G}$ satisfies $x \approx y$. □

Example

The following digraph has Maltsev, join and meet semilattice polymorphisms.



It has only four endomorphisms: id, 0, 1 and inversion, they are all isolated. However, id is connected to 0 among all maps from G to G:

$$\mathrm{id} = x \wedge 1 \to x \wedge a \to x \wedge 0 = 0.$$

Twin polynomials and connectedness on G^G

Definition

Let $\mathbf{A}$ be an algebra. Two unary polynomials $p, q \in \mathrm{Pol}_1(\mathbf{A})$ are **twins** if there exist a term t of arity $n+1$ and constants $\bar{a}, \bar{b} \in A^n$ such that

$$p = t(x, \bar{a}) \quad \text{and} \quad q = t(x, \bar{b}).$$

Let $\sim$ denote the congruence relation on $\mathrm{Pol}_1(\mathbf{A})$ that is the transitive closure of twin polynomials.

Theorem

If a finite algebra $\mathbf{A}$ *has Jónsson terms, then* $\mathrm{id} \sim a$ *for* $a \in A$*.*

Corollary

If a strongly connected [connected and smooth] finite digraph $\mathbb{G}$ *has algebraic length* 1 *and has Jónsson polymorphisms, then* G^G *is strongly connected [connected].*

Algebraic length 1 and Gumm polymorphisms

Lemma

If $\mathbb{G}$ is a digraph and G^G is strongly connected, then $\mathbb{G}$ has a loop.

Proof.

Take $\mathrm{id} \to f_1 \to \cdots \to f_k = c \to \cdots \to f_n \to \mathrm{id}$, then $\mathrm{id} \circ f_1 \circ \cdots \circ f_n \to f_1 \circ \cdots \circ f_n \circ \mathrm{id}$, thus $g = f_1 \circ \cdots \circ f_n$ is a constant endomorphism of $\mathbb{G}$. □

Theorem

If a strongly connected finite digraph $\mathbb{G}$ has algebraic length 1 and has Gumm polymorphisms, then G^G is strongly connected.

Conjecture

If a connected and smooth finite digraph has algebraic length 1 and has Gumm polymorphisms, then it has a near-unanimity polymorphism.

Thank you!